SOME POSSIBLY USEFUL INFORMATION:

$N_A$ or $L = 6.022 \times 10^{23}$ mol$^{-1}$  

$h = 6.626 \times 10^{-34}$ J s

$k = 1.38 \times 10^{-23}$ J K$^{-1}$  

$c = 2.998 \times 10^8$ m s$^{-1}$

H atom mass = $1.66 \times 10^{-27}$ kg  

electron mass = $9.11 \times 10^{-31}$ kg

$R = 8.314$ J K$^{-1}$ mol$^{-1}$

Planck: $E = hv$  

Schroedinger: $\hat{H}\psi = E\psi$  

de Broglie: $\lambda = h/p$

x operator is multiply by $x$  

p operator is $h/(2\pi i) d/dx$

$E_k$ operator is $-h^2/(8\pi^2 m)$ $d^2/dx^2$  

V operator is multiply by $V$

$\sin^2 \theta = (1 - \cos 2\theta)/2$  

$\cos^2 \theta = (1 + \cos 2\theta)/2$
1) **40 points**

a) A particle of mass $m$ is confined by infinite potential barriers to the one-dimensional region of space $x = 0$ to $x = L$, where the potential energy is zero. Its wavefunction is given by

$$\psi(x) = A \sin \left(\frac{n \pi x}{L}\right)$$

where $A$ is a constant. The potential energy is zero for $0 < x < L$.

Verify that this $\psi$ satisfies the Schrödinger equation and find the total energy $E$ of the particle. What is the lowest value $E$ can take?

b) An electron has the wavefunction $\psi = B x^2$ over the range $x = 1$ m to $x = 3$ m, where the potential energy is zero and $B$ is a constant. Elsewhere $\psi = 0$. Find what value of $B$ normalizes the wavefunction.

\[ \begin{align*}
\hat{H} &= \hat{E} + \hat{V} = -\frac{\hbar^2}{8m^2} \frac{d^2}{dx^2} \\
\hat{H} \psi &= -\frac{\hbar^2}{8m^2} \frac{d^2}{dx^2} A \sin \left(\frac{n \pi x}{L}\right) = \frac{\hbar^2}{8mL^2} A n^2 \pi^2 \sin \left(\frac{n \pi x}{L}\right) \\
\text{so } \hat{H} \psi &= E \psi \text{ with } E = \frac{\hbar^2}{8mL^2} n^2 \pi^2 \\
\text{Lowest possible } E \text{ is when } n = 1 \text{ i.e. } \frac{\hbar^2}{8mL^2}.
\end{align*} \]

b) Normalization condition is $1 = \int_{\text{all space}} x^2 \psi^2 dx$

\[ \begin{align*}
\int_1^3 (Bx^2)^2 dx &= B^2 \int_1^3 x^4 dx \\
&= \frac{B^2}{5} \left[x^5\right]_1^3 = \frac{B^2}{5} (3^5 - 1) = B^2 \cdot \frac{27}{5} \\
\therefore B &= \sqrt{\frac{5}{27}} \approx 0.144
\end{align*} \]
2) 30 points

a) Do the position and linear momentum operators commute? What are the implications for the observables of position and linear momentum?

b) A particle is described by the wavefunction \( \psi = e^{ikx} \). Is this wavefunction an eigenfunction of the momentum operator? What can be learned about the observed momentum for this wavefunction?

a) \[ \hat{x} \hat{p} \psi = x \cdot \frac{\hbar}{2\pi i} \frac{d}{dx} \psi \]
\[ \hat{p} \hat{x} \psi = \frac{\hbar}{2\pi i} \frac{d}{dx} (x \psi) = \frac{\hbar}{2\pi i} (\psi + x \frac{d}{dx} \psi) \]

Not equal \( \Rightarrow \) do not commute.

This means \( x \) and \( p \) are complementary observables and there is an uncertainty relationship between them, i.e., \( x \) and \( p \) cannot be found together, simultaneously and exactly.

b) \[ \hat{p} \psi = \frac{\hbar}{2\pi i} \frac{d}{dx} (e^{ikx}) = \frac{\hbar}{2\pi i} ik e^{ikx} = \frac{\hbar k}{2\pi} e^{ikx} = \frac{\hbar k}{2\pi} \psi \]

so yes, \( \psi \) is an eigenfunction and the observed momentum is the eigenvalue, \( p = \frac{\hbar k}{2\pi} \), exactly.
3) 30 points
A photon of wavelength 200 nm knocks an electron out of a piece of metal whose work function is 150 kJ mol$^{-1}$. Calculate the de Broglie wavelength of this electron.

\[ \text{Photon frequency} \quad \nu = \frac{c}{\lambda} = \frac{2.998 \times 10^8 \text{ m s}^{-1}}{200 \times 10^{-9} \text{ m}} = 1.499 \times 10^5 \text{ Hz} \]

\[ \text{Photon energy} \quad E = h \nu = 6.626 \times 10^{-34} \text{ Js} \times 1.499 \times 10^5 \text{ Hz} = 9.93 \times 10^{-19} \text{ J} \]

\[ \text{Work function} \quad \phi = 150 \text{ kJ mol}^{-1} = \frac{150 \times 10^3 \text{ J mol}^{-1}}{6.022 \times 10^{23} \text{ mol}^{-1}} = 2.49 \times 10^{-19} \text{ J} \]

\[ E_k \text{ is the difference} \quad (9.93 - 2.49) \times 10^{-19} \text{ J} = 7.44 \times 10^{-19} \text{ J} \]

\[ E_k = \frac{p^2}{2m} \]

\[ p = \sqrt{2mE_k} = \sqrt{(2 \times 9.11 \times 10^{-31} \text{ kg} \times 7.44 \times 10^{-19} \text{ J})} \]

\[ p = 1.16 \times 10^{-24} \text{ kg m s}^{-1} \]

\[ \text{De Broglie wavelength} \quad \lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ Js}}{1.16 \times 10^{-24} \text{ kg m s}^{-1}} = 5.69 \times 10^{-10} \text{ m} \]