It is important to write neatly and clearly, and to show all working. Allocate time to each question in proportion to the available credit. Keep explanations brief and to the point.

Your name: SOLUTIONS

SOME POSSIBLY USEFUL INFORMATION:

\[ N_A \text{ or } L = 6.022 \times 10^{23} \text{ mol}^{-1} \]
\[ k = 1.38 \times 10^{-23} \text{ J K}^{-1} \]
\[ \text{H atom mass} = 1.66 \times 10^{-27} \text{ kg} \]
\[ R = 8.314 \text{ J K}^{-1} \text{ mol}^{-1} \]
\[ \hbar = 6.626 \times 10^{-34} \text{ J s} \]
\[ c = 2.998 \times 10^8 \text{ m s}^{-1} \]
\[ \text{electron mass} = 9.11 \times 10^{-31} \text{ kg} \]
\[ k_B = 1.38 \times 10^{-23} \text{ J} \]

Planck: \[ E = \hbar v \]
Schrodinger: \[ \hat{H} \psi = E \psi \]
debroglie: \[ \lambda = \hbar / p \]

Particle in 1-D box: \[ \psi_n = (2/a)^{1/2} \sin(n \pi x / a), \quad E_n = \hbar^2 n^2 / (8ma^2) \]

Harmonic oscillator: \[ E_v = (v + 1/2) \hbar \nu; \quad \nu = (k/\mu)^{1/2} / (2\pi) \]

Rigid rotor \[ |L^2| = l(l + 1) \hbar^2 / (4\pi^2) \]
\[ L_z = m \hbar / (2\pi) \]
\[ E = L^2 / (2I) \]

x operator is multiply by x
p operator is \[ \hbar / (2\pi) \text{ d/dx} \]

E_K operator is \[ -\hbar^2 / (8\pi^2 m) \text{ d}^2 / \text{d}x^2 \]
V operator is multiply by V

\[ \sin^2 \theta = (1 - \cos 2\theta) / 2 \]
\[ \cos^2 \theta = (1 + \cos 2\theta) / 2 \]
1) 60 points
In one dimension, a particle is confined to $0 \leq x \leq L$ by infinite potential energy barriers. The potential is zero over $0 \leq x \leq L$, and over this region the wavefunction is

$$\psi(x) = A \sin\left(\frac{5\pi x}{L}\right)$$

$\psi(x) = 0$ elsewhere.

a) Solve the Schrodinger equation to prove that the total energy is $E = \frac{\hbar^2}{8mL^2}$.
b) Derive the value of $A$ which normalizes the wavefunction.
c) Sketch plots of (i) $\psi$ and (ii) $\psi^*\psi$ vs $x$.
d) Based on your plots, or otherwise, deduce the values of $x$ for which the probability density is maximized.
e) If the particle is an electron confined to a region $2 \times 10^{-10}$ m wide, deduce the wavelength of a photon whose energy corresponds to the transition to the next highest energy level.
f) Imagine that for $x < 0$ and $x > L$ the potential energy becomes large but not infinite. Sketch a plot of $\psi$ vs $x$ in this new situation.

d) Because $V=0$, $\hat{H} = \hat{E}$, $\hat{E}\psi = -\frac{\hbar^2}{8\pi^2m} \frac{d^2}{dx^2} A \sin\left(\frac{5\pi x}{L}\right)$

$$= -\frac{\hbar^2}{8\pi^2m} A \cdot \frac{25\pi^2}{L^2} \sin\left(\frac{5\pi x}{L}\right)$$

$$= \frac{25\hbar^2}{8mL^2} \psi = E\psi$$

b) We want $\langle x | x \rangle = \int \psi^* \psi dx = \int_0^L A^2 \sin^2\left(\frac{5\pi x}{L}\right) dx$.

Substitute $\theta = \frac{5\pi x}{L}$, $dx = \frac{L}{5\pi} d\theta$. When $x = L$, $\theta = \frac{5\pi}{2}$, so

$$1 = \int_0^{\frac{5\pi}{2}} A^2 \sin^2\theta \cdot \frac{L}{5\pi} d\theta = \frac{L}{5\pi} A^2 \int_0^{\frac{5\pi}{2}} (1 - \cos 2\theta) d\theta$$

$$= \frac{L\lambda^2}{10\pi} \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{5\pi}{2}} = \frac{L\lambda^2}{10\pi} \left[ \frac{5\pi}{2} - 0 - 0 \cdot \frac{\pi}{2} \right] = \frac{L\lambda^2}{2} \implies A = \sqrt{\frac{2}{L}}.$$
e) \[ m = 9.11 \times 10^{-31} \text{ kg}, \quad L = 2 \times 10^{-10} \text{ m} \]

The next highest level has \( y = A \sin \left( \frac{6\pi x}{L} \right) \) and therefore \( E = \frac{36 \hbar^2}{8mL^2} \).

The energy of the transition is \( \frac{(36-25)\hbar^2}{8mL^2} = \frac{11\hbar^2}{8mL^2} \)

\[ = 11 \times \frac{(6.626 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (2 \times 10^{-10})^2} \]

\[ J = 1.657 \times 10^{-17} \text{ J} \]

\[ \therefore \gamma = 2.500 \times 10^{16} \text{ Hz} \]

\[ \gamma = c/\nu = 1.20 \times 10^{8} \text{ m} \]

f)

\[ y \]

\[ \text{tunneling} \]

\[ L \]

\[ \text{tunneling} \]
2) 20 points

a) Consider the wavefunction \( \psi = A \sin(kx) + A' i \cos(kx) \). Give as much information about the linear momentum as possible.

b) Do the position and momentum operators commute? Show work. What is the physical consequence of this result?

\[ \hat{p} \psi = \frac{\hbar}{2\pi i} \frac{d}{dx} \left( A \sin(kx) + A' i \cos(kx) \right) \]

\[ = \frac{\hbar}{2\pi i} \cdot A \cdot \left( k \cos(kx) - i k \sin(kx) \right) \]

\[ = \frac{\hbar k A}{2\pi i}( -i \cdot i \cos(kx) - i \sin(kx) ) \]

\[ = \frac{\hbar k A}{2\pi i}( -i \cos(kx) - \sin(kx) ) \]

\[ = -\frac{\hbar k}{2\pi} \psi \quad \text{so the momentum is exactly} \quad -\frac{\hbar k}{2\pi} \]

b) \( \hat{x} \hat{p} \psi = x \cdot \frac{\hbar}{2\pi i} \frac{dy}{dx} \psi \)

\[ \hat{p} \hat{x} \psi = \frac{\hbar}{2\pi i} \frac{d}{dx} (xy) = \frac{\hbar}{2\pi i} \left( y + x \frac{dy}{dx} \right) \]

Not equal so \( \hat{x}, \hat{p} \) do not commute.

The consequence is that there is an uncertainty relation between \( x \) and \( p \) and we cannot simultaneously obtain exact information about both. The two observables are said to be "complementary".
3) **20 points**

Light strikes a piece of metal and electrons are emitted with a de Broglie wavelength of $1.3 \times 10^{-9}$ m. If the work function of the metal is $4 \times 10^{-19}$ J, what is the wavelength of the incident light?

\[
E(\text{photon}) = \Phi + E(e^-) \quad \uparrow \text{work function.}
\]

\[
de \text{Broglie} \quad \lambda = \frac{h}{p} \quad \therefore \quad p(e^-) = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34} \text{J}\cdot\text{s}}{1.3 \times 10^{-9} \text{m}} = 5.0 \times 10^{-25} \text{ kg m/s.}
\]

\[
KE = \frac{p^2}{2m} = \frac{(5.10 \times 10^{-25})^2}{2 \times 9.10 \times 10^{-31}} \text{ J} = 1.43 \times 10^{-19} \text{ J.}
\]

\[
\text{Phonon energy} = \Phi + KE = (4 \times 10^{-19} + 1.43 \times 10^{-19}) \text{ J}
\]
\[= 5.43 \times 10^{-19} \text{ J.}
\]

\[
\therefore \quad \nu = \frac{\Phi}{KE} = 8.19 \times 10^{14} \text{ s}^{-1}
\]
\[= \frac{c}{\lambda},
\]
\[\therefore \quad \lambda = \frac{c}{\nu} = \frac{2.998 \times 10^8 \text{ m/s}}{8.19 \times 10^{14} \text{ s}^{-1}} = 3.66 \times 10^{-7} \text{ m} = 366 \text{ nm}.\]